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Matrix Calogero–Sutherland Hamiltonians and the multi-dimensional Darboux transformation

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Abstract

The multi-dimensional Darboux transformation and its spectral properties are shown to give rise to matrix generalizations of the Calogero and Sutherland families of Hamiltonians, acting as twisted Hodge Laplacians on k-forms. These matrix Hamiltonians are shown to be exactly solvable when $k = 1$. In the case of the Sutherland Hamiltonian, the corresponding eigenforms are natural generalizations of the Jack polynomials satisfying remarkable orthogonality properties with respect to the Hodge inner product. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The central role played by the Calogero–Sutherland Hamiltonians [1,5,7] in the theory of classical and quantum completely integrable systems gives a strong motivation to the search for exactly solvable matrix Schrödinger operators which could be viewed as true analogues of these Hamiltonians.

In this paper, we pursue one of the possible approaches to this problem, based on the multi-dimensional Darboux transformation that we have recently introduced and studied for twisted Hodge Laplacians on oriented Riemannian manifolds [2]. We will see that the Darboux transformation gives rise to matrix Calogero–Sutherland Hamiltonians which

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are exactly solvable, with orthogonality relations between their eigenfunctions which are of independent significance, as in the scalar case [4]. It is clear that any purely formal approach is unlikely to produce matrix potentials meeting these requirements.

Recall that the basic ingredient of the multi-dimensional Darboux transformation is a geometric generalization of the intertwining relations that underlie the classical Darboux transformation of Sturm–Liouville operators. The multi-dimensional Darboux transformation therefore enables one to construct a family of twisted Hodge Laplacians whose spectra and eigenforms are related in a natural way through the action of twisted differentials and codifferentials. In local coordinates, these twisted Laplacians correspond to matrix Schrödinger operators acting on k-forms. By applying the multi-dimensional Darboux transformation to the Calogero and Sutherland Hamiltonians, we will therefore obtain matrix Schrödinger operators for a system of N particles interacting pairwise on a line or a circle. In the case $k = 1$, which corresponds to an N-by-N matrix Schrödinger operator, we will derive explicit expressions for the spectrum and the eigenforms of these matrix Schrödinger operators. These eigenforms will be expressible as 1-form analogues of the Laguerre and Jack polynomials, which will be shown to satisfy remarkable orthogonality properties with respect to the Hodge inner product, generalizing the well-known scalar ones.

Section 2 contains a brief summary of the essentials of the multi-dimensional Darboux transformation. In Section 3, we recall the explicit formulas for the bound states and eigenvalues of the Calogero and Sutherland Hamiltonians in terms of Laguerre and Jack polynomials. In Sections 4 and 5, we construct the twisted Hodge Laplacians which arise from the application of the multi-dimensional Darboux transformation to the Calogero and Sutherland models. The corresponding spectral problems are shown to be exactly solvable for $k = 1$, and their eigenforms are computed explicitly together with their orthogonality relations.

2. The multi-dimensional Darboux transformation

Our goal in this section is to recall the essentials of the multi-dimensional generalization of the Darboux transformation to twisted Hodge Laplacians. We refer the reader to [2] for additional details. Let (M, g) be a compact n-dimensional oriented Riemannian manifold without boundary. The exterior algebra $\wedge(M) = \oplus_{k=0}^{n} \wedge^k (M)$ of smooth differential forms on M is endowed with the standard Hodge inner product:

$$(\omega_1, \omega_2) = \int_M \omega_1 \wedge *\omega_2$$

for all $\omega_1, \omega_2 \in \wedge^k(M)$. Our sign convention for the Hodge Laplacian $\Delta_k : \wedge^k(M) \to \wedge^k(M)$ is given by

$$-\Delta_k = d\delta + \delta d, \tag{1}$$

where, for $\omega \in \wedge^k(M)$, we have

$$\delta\omega = (-1)^{n(k-1)+1} * d(*\omega).$$

We consider the *twisted Hodge Laplacians* $H_k : \wedge^k(M) \to \wedge^k(M)$ given by

$$H_k = d^-\delta^+ + \delta^+ d^-, \tag{2}$$

where

$$d^- = e^{-\chi} d\, e^{\chi}, \quad \delta^+ = e^{\chi}\delta\, e^{-\chi}, \tag{3}$$

and χ is a C^∞ real-valued function on M. When $\chi \equiv 0$ we recover the usual Hodge Laplacians $-\Delta_k = d\delta + \delta d$. The twisted differentials d^- and δ^+ act as boundary and coboundary operators on the exterior algebra $\wedge(M)$ and the correponding differential complexes are locally exact. We also have

$$(\delta^+\alpha, \beta) = (\alpha, d^-\beta) \tag{4}$$

for all $\alpha \in \wedge^{k+1}(M)$, $\beta \in \wedge^k(M)$, so that the twisted Hodge Laplacians H_k are self-adjoint and non-negative. When acting on a 0-form, the twisted Hodge Laplacian takes the form of a Schrödinger operator:

$$H_0 = -\Delta_{\mathrm{LB}} + V, \tag{5}$$

where $\Delta_{\mathrm{LB}} = \Delta_0$ denotes the Laplace–Beltrami operator on (M, g), and V is a potential given by

$$V = (\nabla\chi)^2 - (\Delta_{\mathrm{LB}})\chi. \tag{6}$$

The twisted Hodge Laplacians $H_k, k \geq 1$, act as matrix Schrödinger operators when they are expressed in local coordinates. We have

$$H_k = -\Delta_k + V_k, \tag{7}$$

where

$$(V_k\omega)_{i_1,\dots,i_k} = V\omega_{i_1,\dots,i_k} + 2\sum_{r=1}^{k} \nabla^j \nabla_{i_r}\chi . \omega_{i_1,\dots,i_{r-1} j i_{r+1},\dots,i_k}. \tag{8}$$

The twisted Hodge Laplacians H_k can be decomposed as follows:

$$H_k = H_k^{(1)} + H_k^{(2)},$$

where

$$H_k^{(1)} = d^-\delta^+, \quad H_k^{(2)} = \delta^+ d^-.$$

It thus follows that we have the identities:

$$H_k^{(1)} H_k^{(2)} = H_k^{(2)} H_k^{(1)} = 0, \tag{9}$$

and the intertwining relations

$$\delta^+ H_{k+1}^{(1)} = H_k^{(2)}\delta^+, \quad H_{k+1}^{(1)} d^- = d^- H_k^{(2)}.$$

From (9), we obtain the following result:

Lemma 1. *If $\omega \in \wedge^k(M)$ is an eigenform of H_k with eigenvalue $\lambda \neq 0$, then we have either:*
(i) $H_k^{(i)}\omega = \lambda\omega$ *and* $H_k^{(j)}\omega = 0$ *for some* $i, j \in \{1, 2\}$, *or,*
(ii) $H_k^{(i)}\omega$ *is an eigenform of* $H_k^{(i)}$ *with eigenvalue* λ *for* $i = 1, 2$.

We are now ready to define the multi-dimensional Darboux transformation.

Definition 1. Let ω be an eigenform of a twisted Laplacian H_k with eigenvalue $\lambda \neq 0$. If $H_k^{(1)}\omega = \lambda\omega, k \geq 1$, we define its Darboux transform to be the $(k-1)$-form $\delta^+\omega$. If $H_k^{(2)}\omega = \lambda\omega, k \leq n-1$, we define its Darboux transform to be the $(k+1)$-form $d^-\omega$.

The Darboux transformation therefore gives rise to up to three new eigenforms of $H = \oplus_{k=0}^n H_k$, starting from a given eigenform of H_k with non-zero eigenvalue.

Theorem 1. *Suppose that $H_k\omega = \lambda\omega, \lambda \neq 0$. In* case (i) *of Lemma* 1 *with $i = 1$, the Darboux transform $\delta^+\omega$ is an eigenform of H_{k-1} with eigenvalue λ. In* case (i) *of Lemma* 1 *with $i = 2$, the Darboux transform $d^-\omega$ is an eigenform of H_{k+1} with eigenvalue λ. In* case (ii) *of Lemma* 1, *$\delta^+ H_k^{(1)}\omega$ and $d^- H_k^{(2)}\omega$ are eigenforms of H_{k-1} and H_{k+1} with eigenvalue λ, respectively, and $H_k^{(1)}\omega$, $H_k^{(2)}\omega$ are eigenforms of H_k with eigenvalue λ.*

The operator $H = \oplus_{k=0}^n H_k$ can thus be thought of as a supersymmetric Hamiltonian acting on the exterior algebra of M [8]. We conclude by remarking that the multi-dimensional Darboux transformation maps any pair of orthogonal eigenfunctions corresponding to distinct eigenvalues of a twisted Laplace–Beltrami operator H_0 to orthogonal eigen-1-forms of the corresponding twisted Hodge Laplacian H_1.

Theorem 2. *Let $H = d^-\delta^+ + \delta^+ d^-$ be a twisted Laplacian and let α, β be 0-forms such that $H\alpha = \lambda\alpha$, $H\beta = \mu\beta$ where $\lambda \neq \mu$ and $(\alpha, \beta) = 0$. Then,*

$$(d^-\alpha, d^-\beta) = 0.$$

Proof. Let H be a twisted Laplacian acting on 0-forms. It follows from (4) that

$$\begin{aligned}(d^-\alpha, d^-\beta) &= (\alpha, \delta^+ d^-\beta) = (\alpha, H\beta) = \mu(\alpha, \beta) = 0 \\ &= (d^-\alpha, d^-\beta) = (\delta^+ d^-\alpha, \beta) = (H\alpha, \beta) = \lambda(\alpha, \beta). \qquad \square\end{aligned}$$

In Sections 4 and 5, we will apply Theorems 1 and 2 to obtain exactly solvable matrix generalizations of the Calogero–Sutherland class of Hamiltonians.

3. The Calogero and Sutherland Hamiltonians

The Calogero Hamiltonian is given by

$$H_{\mathrm{C}} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=2}^{N}\sum_{j=1}^{i-1} \left\{ \frac{1}{8}\omega^2 (x_i - x_j)^2 + g(x_i - x_j)^{-2} \right\}, \tag{10}$$

where g is a constant satisfying $g > -1/2$ to ensure the existence of bound states. The Hamiltonian (10) is thus of the form (5) where M is the open subset of $\mathbb{R}^N$ given by $x_1 < \cdots < x_N$, endowed with the flat Euclidean metric. In terms of the variables z and r^2 given by

$$z = \prod_{i=2}^{N}\prod_{j=1}^{i-1} (x_i - x_j), \qquad r^2 = \frac{1}{N}\sum_{i=2}^{N}\sum_{j=1}^{i-1} (x_i - x_j)^2,$$

the eigenfunctions of H_{C} are given by

$$\psi_{nl}(\mathbf{x}) = z^{a+1/2} \varphi_{nl}(r) P_l(\mathbf{x}), \quad n, l = 0, 1, 2, \ldots,$$

where

$$\begin{aligned} \varphi_{nl}(r) &= \exp\left[-\left(\frac{\omega}{4}\right)\left(\frac{N}{2}\right)^{1/2} r^2 \right] L_n^b \left(\frac{\omega}{2}\left(\frac{N}{2}\right)^{1/2} r^2 \right), \\ a &= \frac{1}{2}(1+2g)^{1/2}, \\ b &= l + \frac{1}{2}(N-3) + \frac{1}{2}N(N-1)\left(a + \frac{1}{2}\right). \end{aligned} \tag{11}$$

The functions L_n^b are the Laguerre polynomials and the functions $P_l(\mathbf{x})$ are symmetric harmonic polynomials of degree l in the particle coordinates with respect to a certain generalized Laplacian [1]. The eigenvalues of the Hamiltonian H_{C} are given by

$$E_{2n+l} = \omega \left(\frac{N}{2}\right)^{1/2} \left[\frac{1}{2}(N-1) + \frac{1}{2}N(N-1)\left(a + \frac{1}{2}\right) + 2n + l \right]. \tag{12}$$

The Sutherland Hamiltonian describes a system of N particles on a circle, interacting through a pairwise potential. It is given by

$$H_{\mathrm{S}} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\beta(\beta - 1) \sum_{i<j} \frac{1}{(L/\pi)^2 \sin^2((\pi/L)(x_i - x_j))}, \tag{13}$$

where $0 < x_1 < \cdots < x_N < L$, $\beta(\beta - 1) \geq -1/4$ and L denotes the perimeter of the circle, so that $(L/\pi)\sin((\pi/L)(x_i - x_j))$ is the chord length between the i and j particles. The spectrum of the Sutherland Hamiltonian is bounded from below, with ground state energy given by

$$E_0 = \frac{1}{3}\left(\frac{\pi}{L}\right)^2 \beta^2 N(N^2 - 1). \tag{14}$$

The normalizable ground state is given by

$$\psi_0(x_1, \dots, x_N) = \prod_{1 \le i < j \le N} \left[\sin\left(\frac{\pi}{L}(x_i - x_j)\right)\right]^{\beta}. \tag{15}$$

It is convenient to transform the original spectral problem into a related one by conjugating H_S by the normalizable ground state eigenfunction ψ_0 and introducing the variables

$$z_j = e^{2\pi i x_j / L}.$$

The spectral problem $H_S \psi = E\psi$ is thus transformed into

$$H\phi = \left(\frac{L}{2\pi}\right)^2 (E - E_0)\phi,$$

where

$$H = \sum_{j=1}^{N} \left(z_j \frac{\partial}{\partial z_j}\right)^2 + \beta \sum_{j<k} \left(\frac{z_j + z_k}{z_j - z_k}\right)\left(z_j \frac{\partial}{\partial z_j} - z_k \frac{\partial}{\partial z_k}\right). \tag{16}$$

The eigenfunctions of H are certain polynomials $J_\lambda(z_1, \dots, z_N; 1/\beta)$ of the z_j's which are labelled by partitions λ of their degree n of length less than N. Recall that if λ and μ are two partitions of n, then we have a natural ordering given by $\lambda \ge \mu$ if $\lambda_1 + \lambda_2 + \cdots + \lambda_i \ge \mu_1 + \mu_2 + \cdots \mu_i$, for all i. The eigenfunctions $J_\lambda(z_1, \dots, z_N; 1/\beta)$ of H will be of the form

$$J_\lambda(z_1, \dots, z_N; 1/\beta) = m_\lambda + \sum_{\mu < \lambda} v_{\lambda\mu} m_\mu,$$

where

$$m_\lambda = \sum_{\mathrm{perm}(\lambda)} \prod_j z_j^{\lambda_j}$$

and they will be orthonormal with respect to the inner product given by

$$(J_\lambda, J_\mu) = \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi} \prod_{j<k} |(z_j - z_k)|^{2\beta} J_\lambda(\mathbf{z}) J_\mu(\mathbf{z}). \tag{17}$$

These conditions determine uniquely the Jack polynomials [4]. The excited states are expressed in terms of the Jack polynomials in the following way:

$$\psi_{\lambda,q}(z_1, \dots, z_N) = \left(\prod_{i=1}^{N} z_i\right)^{q-(N-1)\beta/2} \prod_{i<j} (z_i - z_j)^{\beta} J_\lambda(z_1, \dots, z_N; 1/\beta), \tag{18}$$

where q is an arbitrary real quantum number corresponding to the translational invariance of the Hamiltonian and λ covers all partitions of length less than N. The eigenvalues $\kappa_{\lambda,q}$

and $E_{\lambda,q}$ of the Sutherland Hamiltonian H_S and the momentum operator P acting on the eigenstates (18) are given by

$$\kappa_{\lambda,q} = \frac{2\pi}{L}(n + Nq) = \sum_{i=1}^{N} \kappa_i,$$
$$E_{\lambda,q} = \sum_{i=1}^{N} \kappa_i^2, \tag{19}$$

where

$$\kappa_i = \frac{2\pi}{L}[\lambda_i + \beta(N + 1 - 2i) + q].$$

4. Matrix Calogero potentials and their eigenforms

In this section, we construct an exactly solvable matrix generalization of the Calogero Hamiltonian, realized as a twisted Hodge Laplacian acting on 1-forms. The eigen 1-forms and the spectrum of this matrix Hamiltonian will be determined explicitly by applying Theorem 1. To define the twisted Hamiltonian H, we take $e^{-\chi}$ as a constant multiple of the ground state, ψ_0, since this eigenfunction has no zeros away from the singularities in the potential. We thus let

$$\chi(\mathbf{x}) = Cr^2 - \left(a + \frac{1}{2}\right)\log(z), \tag{20}$$

where

$$C = \left(\frac{\omega}{4}\right)\left(\frac{N}{2}\right)^{1/2}.$$

We have

$$\frac{\partial \chi}{\partial x_i} = \frac{2C}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}(x_i - x_j) - \left(a + \frac{1}{2}\right)\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{(x_i - x_j)}$$

and

$$\Delta\chi = 2C(N-1) + \left(a + \frac{1}{2}\right)\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{(x_i - x_j)^2}.$$

The Laplacian of χ can be written in an equivalent form by using the identity

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{(x_i - x_j)^2} = 2\sum_{i=2}^{N}\sum_{j=1}^{i-1}\frac{1}{(x_i - x_j)^2},$$

which gives

$$\Delta\chi = 2C(N-1) + 2\left(a+\frac{1}{2}\right)\sum_{i=2}^{N}\sum_{j=1}^{i-1}\frac{1}{(x_i-x_j)^2}.$$

Using Eqs. (7) and (8), we obtain

$$\begin{aligned}(H_{\mathrm{C}k}\omega)_{i_1\ldots i_k} = \Bigg[&-\Delta + \sum_{i=1}^{N}\left(\frac{2C}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}(x_i-x_j) - \left(a+\frac{1}{2}\right)\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{(x_i-x_j)}\right)^2 \\ &-2C(N-1) + 2\left(a+\frac{1}{2}\right)\sum_{i=2}^{N}\sum_{j=1}^{i-1}\frac{1}{(x_i-x_j)^2}\Bigg]\omega_{i_1,\ldots,i_k} \\ &+2\sum_{r=1}^{k}\sum_{\substack{j=1\\ j\neq i_r}}^{N}\left(-\frac{2C}{N} - \left(a+\frac{1}{2}\right)\frac{1}{(x_j-x_{i_r})^2}\right)\omega_{i_1,\ldots,i_{r-1}ji_{r+1},\ldots,i_k} \\ &+2\sum_{r=1}^{k}\left(\frac{2C}{N}(N-1) + \left(a+\frac{1}{2}\right)\sum_{\substack{q=1\\ q\neq i_r}}^{N}\frac{1}{(x_{i_r}-x_q)^2}\right)\omega_{i_1,\ldots,i_r,\ldots,i_k}.\end{aligned} \tag{21}$$

For the twisted Laplacian $H_{\mathrm{C}1}$ acting on 1-forms, we thus have

$$\begin{aligned}(H_{\mathrm{C}1}\omega)_m = \Bigg[&-\Delta + \sum_{i=1}^{N}\left(\frac{2C}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}(x_i-x_j) - \left(a+\frac{1}{2}\right)\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{(x_i-x_j)}\right)^2 \\ &-2C(N-1) + 2\left(a+\frac{1}{2}\right)\sum_{i=2}^{N}\sum_{j=1}^{i-1}\frac{1}{(x_i-x_j)^2}\Bigg]\omega_m \\ &-2\sum_{\substack{j=1\\ j\neq m}}^{N}\left(\frac{2C}{N} + \left(a+\frac{1}{2}\right)\frac{1}{(x_m-x_j)^2}\right)\omega_j \\ &+2\left(\frac{2C}{N}(N-1) + \left(a+\frac{1}{2}\right)\sum_{\substack{q=1\\ q\neq m}}^{N}\frac{1}{(x_m-x_q)^2}\right)\omega_m.\end{aligned} \tag{22}$$

The eigenforms of $H_{\mathrm{C}1}$ can now be computed by applying d^- to the eigenfunctions of H_{C}. Indeed, recall from Theorem 1 that if ψ is an eigenfunction of H_0 with eigenvalue $E \neq E_0$, then $\omega = d^-\psi$ is an eigenform of H_1 with eigenvalue $E - E_0$. We have

$$(d^-\psi)_j = q_j^-\psi,$$

where

$$q_j^- = e^{-\chi}\nabla_j e^{\chi}.$$

From (10), we obtain

$$q_j^- \psi_{nl} = e^{-\chi} \nabla_j e^{\chi} \psi_{nl} = z^{a+1/2} e^{-Cr^2} \partial_j (L_n^b(2Cr^2) P_l(\mathbf{x})).$$

Therefore, the eigen 1-forms of H_{C1} are given as follow.

$$\omega_{nl} = d^- \psi_{nl} = z^{a+1/2} e^{-Cr^2} \sum_{j=1}^{N} \partial_j \{L_n^b(2Cr^2) P_l(\mathbf{x})\} \, \mathrm{d}x_j. \tag{23}$$

The latter expression can be made more explicit by using the recursion formula for the Laguerre polynomials. We have

$$\omega_{nl} = (d^- \psi_{nl}) = -\frac{4C}{N} z^{a+1/2} e^{-Cr^2} \times \sum_{j=1}^{N} \left(\sum_{\substack{s=1 \\ s \neq j}}^{N} (x_j - x_s) L_{n-1}^{b+1}(2Cr^2) P_l(\mathbf{x}) + L_n^b(2Cr^2) \partial_j P_l(\mathbf{x}) \right) \mathrm{d}x_j. \tag{24}$$

Using Theorems 1 and 2, we can summarize the results of this section as follows:

Theorem 3. *The 1-forms ω_{nl} defined by* (24) *are the eigenforms of the matrix Calogero Hamiltonian H_{C1} given by* (22), *with eigenvalues E_{nl} given by* (12). *The eigenforms ω_{nl} and $\omega_{n'l'}$ corresponding to distinct eigenvalues are orthogonal with respect to the Hodge inner product,*

$$\int \omega_{nl} \wedge *\omega_{n'l'} = \delta_{nn'} \delta_{ll'},$$

that is,

$$\int \sum_k z^{2a+1} e^{-2Cr^2} \partial_k [L_n^b(2Cr^2) P_l(\mathbf{x})] \partial_k [L_{n'}^{b'}(2Cr^2) P_{l'}(\mathbf{x})] \, \mathrm{d}x_1 \cdots \mathrm{d}x_N = \delta_{nn'} \delta_{ll'}.$$

5. Matrix Sutherland models and the Jack eigenforms

We now turn to the Sutherland model, in which we let $L = 2\pi$. Just as we did with the Calogero model, we choose $e^{-\chi}$ to be a constant multiple of the ground state,

$$\chi(\mathbf{x}) = -\beta \sum_{1 \le j \le N} \log \sin \frac{1}{2}(x_j - x_i), \tag{25}$$

which can be rewritten as

$$\chi(\mathbf{x}) = -\beta \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \log \sin \frac{1}{2}(x_j - x_i).$$

We have

$$\frac{\partial \chi}{\partial x_i} = -\frac{\beta}{2}\sum_{\substack{j=1\\ j\neq i}}^{N} \cot\frac{1}{2}(x_i - x_j)$$

and

$$\Delta\chi = \frac{\beta}{4}\sum_{s=1}^{N}\sum_{\substack{i=1\\ i\neq s}}^{N}\frac{1}{\sin^2(1/2)(x_s - x_i)}.$$

Therefore, we obtain from (7) and (8) that the twisted Hodge Laplacians arising from the Sutherland model are given by

$$\begin{aligned}(H_{Sk}\omega)_{i_1,\dots,i_k} = {} & \left[-\Delta + \frac{\beta^2}{4}\sum_{s=1}^{N}\left(\sum_{\substack{i=1\\ i\neq s}}^{N}\cot\frac{1}{2}(x_s - x_i)\right)^2 \right.\\ & \left. -\frac{\beta}{4}\sum_{s=1}^{N}\sum_{\substack{i=1\\ i\neq s}}^{N}\frac{1}{\sin^2(1/2)(x_s - x_i)}\right]\omega_{i_1,\dots,i_k}\\ & -\beta\sum_{r=1}^{k}\sum_{\substack{j=1\\ j\neq i_r}}^{N}\frac{1}{\sin^2(1/2)(x_j - x_{i_r})}\omega_{i_1,\dots,i_{r-1}ji_{r+1},\dots,i_k}\\ & +\frac{\beta}{2}\sum_{r=1}^{k}\sum_{\substack{q=1\\ q\neq i_r}}^{N}\frac{1}{\sin^2(1/2)(x_{i_r} - x_q)}\omega_{i_1,\dots,i_r,\dots,i_k}. \end{aligned} \tag{26}$$

The Hamiltonian H_{S1} acting on 1-forms is given by

$$\begin{aligned}(H_{S1}\omega)_m = {} & \left[-\Delta + \frac{\beta^2}{4}\sum_{s=1}^{N}\left(\sum_{\substack{i=1\\ i\neq s}}^{N}\cot\frac{1}{2}(x_s - x_i)\right)^2 \right.\\ & \left. -\frac{\beta}{4}\sum_{s=1}^{N}\sum_{\substack{i=1\\ i\neq s}}^{N}\frac{1}{\sin^2(1/2)(x_s - x_i)}\right]\omega_m\\ & +\frac{\beta}{2}\sum_{\substack{j=1\\ j\neq m}}^{N}\frac{1}{\sin^2(1/2)(x_m - x_j)}\omega_j + \frac{\beta}{2}\sum_{\substack{q=1\\ q\neq m}}^{N}\frac{1}{\sin^2(1/2)(x_m - x_q)}\omega_m. \end{aligned} \tag{27}$$

We now apply the multi-dimensional Darboux transformation to construct a matrix generalization of the Sutherland Hamiltonian, realized as a twisted Hodge Laplacian acting on

1-forms. Just as in the case of the Calogero Hamiltonian, the eigenforms and the spectrum of this matrix Hamiltonian will be determined explicitly by applying Theorem 1. These eigenforms will be expressed in terms of the Jack polynomials and will satisfy remarkable orthogonality properties relative to the Hodge inner product. We will refer to them as *Jack 1-forms* . It is convenient to work in the coordinate chart z_j used in Section 3. In these coordinates, the ground state wavefunction is given by

$$\Delta^{\beta}(\mathbf{z}) = \prod_{j<k} |z_j - z_k|^{\beta} \prod_k z_k^{-\beta(N-1)/2},$$

up to complex multiplicative constants, and the Schrödinger operator is transformed into (16).

We therefore set

$$\chi(\mathbf{x}(\mathbf{z})) = -\log \Delta^{\beta}(\mathbf{z}) = \beta \sum_{j<k} \log |z_j - z_k| + \frac{\beta(N-1)}{2} \sum_k \log(z_k). \tag{28}$$

We have

$$q_k^- \psi_{\lambda,q} = \mathrm{e}^{-\chi} \nabla_k \mathrm{e}^{\chi} \psi_{\lambda,q} = \left(\prod_{m=1}^{N} z_m\right)^{-(N-1)\beta/2} \times \prod_{i<j} |z_i - z_j|^{\beta} \partial_{z_k} \left[\left(\prod_{m=1}^{N} z_m\right)^{q} J_{\lambda}\left(\mathbf{z}; \frac{1}{\beta}\right),\right]$$

so that

$$d^- \psi_{\lambda,q} = \left(\prod_{i=1}^{N} z_i\right)^{-(N-1)\beta/2} \prod_{i<j} |z_i - z_j|^{\beta} \sum_k \partial_{z_k} \left[\left(\prod_{m=1}^{N} z_m\right)^{q} J_{\lambda}\left(\mathbf{z}; \frac{1}{\beta}\right)\right] \mathrm{d}z_k$$

This differential form is a Jack 1-form.

Theorem 4. *The 1-forms given by*

$$\omega_{\lambda,q} := (d^- \psi_{\lambda,q}) = \left(\prod_{i=1}^{N} z_i\right)^{-(N-1)\beta/2} \prod_{i<j} |z_i - z_j|^{\beta} \sum_k \partial_{z_k} \times \left[\left(\prod_{m=1}^{N} z_m\right)^{q} J_{\lambda}\left(\mathbf{z}, \frac{1}{\beta}\right)\right] \mathrm{d}z_k$$

are eigenforms of the matrix Sutherland Hamiltonian H_{S1} *given by* (27) *with eigenvalues* $E_{\lambda,q}$ *given by* (19). *They are orthogonal with respect to the Hodge inner product,*

$$\int \omega_{\lambda,q} \wedge *\omega_{\lambda',q'} = \delta_{\lambda,\lambda'} \delta_{q,q'},$$

that is,

$$\int \sum_k (z_k)^2 \left(\prod_{m=1}^N z_m\right)^{-(N-1)\beta} \prod_{i<j} |z_i - z_j|^{2\beta} \partial_{z_k} \left[\left(\prod_{s=1}^N z_s\right)^q J_\lambda\left(\mathbf{z}; \frac{1}{\beta}\right)\right] \partial_{z_k}$$
$$\left[\left(\prod_{s=1}^N z_s\right)^{q'} J_{\lambda'}\left(\mathbf{z}; \frac{1}{\beta}\right)\right] \left(\prod_{m=1}^N z_m^{-1}\right) \mathrm{d}z_1 \cdots \mathrm{d}z_N = \delta_{\lambda\lambda'}\delta_{q,q'}$$

The orthogonality relations given in Theorems 3 and 4 can be thought of as natural generalizations to the case of differential forms of the classical orthogonality relations for the Laguerre and Jack polynomials. We conclude by remarking that similar results can be obtained for all the families of integrable Hamiltonians related to the simple Lie algebras, in the sense of [5]. The corresponding eigenforms will be expressed in terms of symmetric Heckman–Opdam polynomials [3].

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